USING REAL PROPERTY OPTION VALUATION METHODS WHEN MARKET DATA IS NOT AVAILABLE

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ABSTRACT
Titman (1985) famously applied optionality to derive the value of a vacant block of land then used as a car park at UCLA for which, in common with traditional property valuation approaches, a willing but not anxious buyer was assumed. However, what might the value of such a vacant block of land be in a property market downturn when willing but not anxious buyers are hesitant and there is an absence of comparable sales for reference? This paper applies concepts of optionality and indifference pricing to explore how such a vacant block of land might be considered relative to other, non-property asset classes to derive an assessment of value in the absence of an active property market.

Keywords: real property options, relative pricing, certainty equivalents, indifference pricing, risk measures.

INTRODUCTION
The generally accepted definition of market value is premised on a range of assumptions:

> "Market value is the estimated amount for which an asset should exchange on the valuation date between a willing buyer and a willing seller in an arm's length transaction, after proper marketing and where the parties had each acted knowledgeably, prudently and without compulsion." (IVSC 2011, p20)

being distinguishable from the definitions of price, fair value and worth or investment value.

Such a definition of market value implicitly assumes the existence of an active market. Traditional property valuation approaches adopting such a definition of market value generally rely on information from such an active property market (such as rental rates, capitalisation rates, discount rates and so forth) for use in the property valuation process. Effectively, in an active property market, property is being valued by reference to other property rather than to the wider capital markets.

However, in a property market downturn, an active market may temporarily cease to exist. Transactions may cease to occur for a sustained period while sellers, buyers and lenders wait to see what happens in the capital markets, financial markets and wider property markets. In a property market downturn, lenders may foreclose on distressed borrowers and take control of a property, placing it on the market for sale. Traditional property valuation approaches, being reliant on historic transactions, are deprived of information with which to determine the market value of such a property and so can only determine what it might have been before the downturn and apply some form of deflator to approximate what value might now be as the basis for the lender to consider any offers made.

Similarly, potential buyers relying on property market information may adopt a similar approach, while those relying on an assessment of the property relative to other non-property asset classes (such as cash, equities and bonds) may be able to determine a value at which they would be willing to transact which, if accepted by the lender, may form evidence of market value.

Take, for example, the vacant block of land in Westwood, Los Angeles comprising the UCLA car park lot famously considered by Titman (1985) in his seminal paper on the application of optionality to property valuation. Titman (1985) implicitly assumes an active property market, but what if the Westwood property market was in a downturn and no transactions had occurred for over a year? The absence of transactions does not mean that the car park is of no value, but simply that traditional property valuation methods are deprived of the information required with which to determine market value.

The Westwood car park will transact in the, albeit depressed, local property market at a level of value which a buyer is willing to pay and a seller is willing to accept. Traditional property valuation approaches may be adopted to suggest a value of, say, 20% to 40% below the level of historic transactions but lack any rigorous basis for assessment, such that the lender may be unable to determine whether to accept an offer at -40% as the market may include other buyers at -20% yet to bid.

Similarly, traditional property valuation approaches lack direct relativity to other, non-property asset classes. In the event that the Westwood car park was offered for sale on foreclosure by the lender, there may be hypothesised to be a point at which a diversified investor would conclude that the risk-return expectations from such a property were superior to those from his existing cash, equity or bond investments and so sell such investments to buy the property. In essence, a diversified investor may choose to move from a portfolio of cash, equities and bonds to a portfolio of cash, equities, bonds and car park dependent upon relative risk-return expectations.

This paper seeks to apply concepts of optionality and indifference pricing to explore how such a vacant block of land might be considered relative to other, non-property asset classes to derive an assessment of value in the absence of an active property market. The paper does not seek to prescribe a series of formulae for immediate application by property valuation practitioners but to raise issues for consideration and discussion by property valuation academics and practitioners as part of the evolution of the property valuation discipline.

The next section briefly considers some of the more relevant literature, with the following section then presenting a simple example of an incomplete market. Thereafter, the next section considers the application of concepts of optionality, with the following section considering application of concepts of indifference pricing and the final section comprising conclusions.

REVIEW OF LITERATURE

Titman (1985) applied concepts of optionality to develop a valuation equation for pricing vacant lots in urban areas, adopting a range of fundamental assumptions including there being numerous available lots, an active property market and market participants who were private investors seeking to maximise wealth. Central to Titman's model was construction cost, with the vacant lot viewed as an option to purchase one of a number of different possible buildings at exercise prices that are equal to their respective construction costs. The author sought to explicitly address uncertainty about how future real estate prices might affect current real estate activities, being an issue considered to be inadequately considered in traditional property valuation texts of the time.

Notably, Titman (1985) included property market information (construction costs) and a risk free rate rather than including other capital market or financial market variables in the model, such that this application of optionality was, like traditional property valuation methods, property market based. Geltner (1989) identifies these shortcomings and adds consideration of expected growth in future land values, arbitrage and investor expectations regarding urban land economic dynamics.

Titman (1985) uses a simple binomial approach but makes financial assumptions in order to value optionality associated with a block of land. While at the time this was a significant advance in option pricing in this context, Titman had to make assumptions that can only hold in a limited way in this context.

Other early major treatments of real options and their valuation were provided in monographs by Dixit and Pindyck (1994) and Trigeorgis (1997). Both texts essentially assume that agents (hereinafter referred to as market participants) are risk neutral, that the underlying assets in real options are tradeable and that the payoffs of real options can be replicated by using cash and tradeable securities. While these assumptions can be justified in financial markets, they are at best approximations when it comes to real options and the property asset class. Not all property assets (e.g. the vacant block of land) can be treated in this way and market participants assign values to projects depending upon their risk aversion. Dixit and Pindyck (1994) adopt continuous time modelling generalising the methodology of Black and Scholes, while Trigeorgis (1997) takes the binomial modelling approach.

Other authors have considered optionality in the context of land tenure (Cappoza and Sick, 1991), property investment and leases (Grenadier, 1995) and property development/ abandonment (Williams, 1991). Patel, Paxson and Sing (2005), in their comprehensive RICS Research Paper, considered optionality in the context of six primary practical uses, including property research, investment, leasing, operations, funding and strategy with Patel and Paxson (2001) undertaking an innovative application of optionality to the valuation of Canary Wharf.

However, the issues of property valuation in a property market downturn characterised by a lack of transactions and valuation by reference to non-property asset classes in an optionality context do not appear to have received focused attention within that literature reviewed.

SIMPLE EXAMPLE OF AN INCOMPLETE MARKET

Most approaches to pricing optionality in the context of property valuation adopt techniques for the valuation of financial options. However, for these techniques to be valid, it is often necessary that optionality be derived from a treadable asset that can be bought and sold in any quantity and without loss of value.

Moreover, the payoff from optionality needs to be attainable (or hedgeable) in the sense that it can be recreated using tradeable assets (such as cash, equities or bonds). This is generally not the case with property assets which are usually not homogenous, indivisible, illiquid and slow to trade. Using terminology from the finance discipline, the property market may be described as *not complete*, meaning that it is not necessarily possible to recreate the payoff from a transaction in property (such as a vacant block of land) using other tradeable assets (such as cash, equities or bonds).

For practical applications of optionality, it is therefore important to use a pricing methodology capable of assigning value to optionality even when that optionality is derived from an asset that may not be tradeable, such as property (eg: a vacant block of land) in a market downturn. Further, such methodology should be capable of effective application to all types of assets and should be capable of assigning a correct value to assets that are tradeable.

As will be shown throughout this section, a number of popular approaches to valuing optionality fall short of satisfying such requirements for both tradeable (such as cash, equities or bonds) and non-tradeable (such as property) assets.

In order to illustrate the difficulties with the valuation of optionality derived from non-financial assets such as property, we begin with a simple two-period example of an *incomplete* market which we will then use in subsequent sections to illustrate various pricing methodologies culminating with indifference pricing.

We consider a two-period financial market with times $t = 0$ and $t = 1$. We let S denote a tradeable asset which we assume could represent a matching index (such as a direct property index, rather than a REIT index, measured at t = 0 and at t = 1). At $t = 0$, the value is $S(0)$ and at $t = 1$, it may take three values $S(1, j)$ in three states (such as downturn, stable and boom market states) labelled by j, with probability p_j for $j = 1,2,3$. We use three states in our simple model because it then provides the simplest example of an incomplete market (containing non-attainable claims, being claims that cannot be recreated by trading cash and a property index in the three nominated states) for which we study various valuation principles.

We assume that $p_j > 0$ for $j = 1,2,3$ and $p_1 + p_2 + p_3 = 1$. For the purposes of a worked example, let $S(0) = 800$, $S(1,1) = 640$ and $S(1,2) = 800$, $S(1,3) = 960$ with probabilities $p_1 = 0.30$, $p_2 = 0.50$ and $p_3 = 0.20$.

We assume that the one period risk-free interest rate is 10%, so that 1 at time $t = 0$ becomes $R = 1.1$ in all the three states at time $t = 1$.

Now consider a property asset (G), such as a vacant block of land, which may have a value in each of the three states (such as downturn, stable and boom market). Assume such an asset G with payoff $G(1, j)$, $j = 1, 2, 3$ is *attainable* which means that payoff to G at time t = 1, in all three possible states of the economy, can be recreated by trading cash and a property index at time t = 0. Therefore, there must exist $a, b \in \mathbb{R}$ so that:

$$aR + bS(1, j) = G(1, j) \qquad \text{Equation 1}$$

for $j = 1,2,3$, In this two-period market, the asset G is *attainable* if and only if:

$$G(1,2) = \frac{1}{2}\left[G(1,1) + G(1,3)\right] \qquad \text{Equation 2}$$

which is a necessary and sufficient condition that the three Equations 1 in two unknowns a, b have a solution. As there are many examples of assets with payoffs which do not satisfy Equation 2, this market is *incomplete.*

This means that, in this market, there are positions that cannot be hedged or recreated using cash and the tradeable asset S (the property index). It would not, therefore, be appropriate to apply any valuation methodology that relies on the market being complete, as is the case with many approaches to option valuation in finance and their applications to valuing optionality in the context of property markets.

APPLICATION OF CONCEPTS OF OPTIONALITY

In this section we adopt the notation and setup of the model in the previous sections and consider two approaches to valuing optionality: relative pricing and utility-based pricing.

Relative pricing follows the methodology of Black and Scholes where a payoff of an asset can be recreated by a portfolio of financial assets and the present value of the asset is then the present value of the replicating portfolio. This approach is at the core of the real option pricing methodologies introduced by Dixit and Pindyck (1994) and Trigeorgis (1997). Effectively, the approach assumes that it is always possible to recreate the payoff from a property (such as a vacant block of land, being a non-tradeable asset) by using a property index and cash (being tradeable assets).

Utility-based pricing derives from the seminal work of John von Neumann and Oscar Morgenstern on the applications of expected utility to quantifying preferences over risky alternatives, one of the key concepts in modern economics. Effectively, this approach assumes that faced with either a property (such as a vacant block of land) or a bundle of property index and cash for example, an investor will have a preference which can be represented mathematically using a utility function. If payoffs from investment alternatives are uncertain, for example dependent on different states of the economy (such as downturn, stable and boom market), then the probabilities associated with these states are used to calculate the expected utility. The objective of investing optimally is then formulated in terms of maximising expected utility.

We shall use our simple example of an incomplete market from the previous section to illustrate the difficulties that arise when these two approaches are applied to claims that are not attainable (such as a vacant block of land).

Relative Pricing

The key assumption in applications of a relative pricing approach in the context of property valuation is that the payoff from optionality associated with a property asset (such as a vacant block of land) can be recreated using a portfolio of traded assets (such as a property index and cash).

In the simple incomplete market of the previous section, first suppose that G, which could represent optionality embedded in a property asset such as a vacant block of land, is attainable – which means that Equation 2 holds – then Equation 1 holds with:

$$a = \frac{5G(1,1) - 4G(1,2)}{1.1} \qquad \text{Equation 3}$$

$$b = \frac{G(1,2) - G(1,1)}{160}. \qquad \text{Equation 4}$$

Suppose that S is tradeable (such as a property index), so that it can be purchased in any quantities. As there are no arbitrage opportunities in the market of the previous section, the law of one price holds, and:

$$G(0) = a + bS(0) = -\frac{5}{11}G(1,1) + \frac{15}{11}G(1,2) = \frac{5}{22}G(1,1) + \frac{15}{22}G(1,3) \qquad \text{Equation 5}$$

where the first expression follows from Equations 3 and 4, and the second uses Equation 2. This second expression makes it clear that if the payoff of an attainable claim is positive, then so is the present value.

If G has the payoffs:

$$G(1,1) = 80, \quad G(1,2) = 96, \quad G(1,3) = 128$$

then G is not attainable and there is not a clear price for G at time $t = 0$.

We could find the value at $t = 0$ of the lowest cost portfolio that will generate not less than the cash flow of asset G. This minimal *super-replicating portfolio* represents the cheapest way of guaranteeing that the payoff to G can be met by the seller. This means that we minimize:

$$a + b800$$

subject to:

$$aR + bS(1, j) \geq G(1, j)$$

for $j = 1,2,3$. This is a linear programming problem:

$$\text{minimize} \quad a + b800$$

subject to

$$\begin{aligned} 1.1a + 960b &\geq 128 \\ 1.1a + 800b &\geq 96 \\ 1.1a + 640b &\geq 80. \end{aligned}$$

The minimum value of the objective function is:

$$\frac{1160}{11} = 105.4545\ldots$$

with

$$\hat{a} = -\frac{160}{11}$$

and

$$\hat{b} = \frac{3}{20}$$

being the optimizers. This is the *asking price* for G being, effectively, the minimum acceptable price to the seller of the vacant block of land.

A transaction will only occur where potential buyers are willing to pay at least the asking price or more as a bid price.

The *bid price* of G, being the maximum potential amount that a buyer of the vacant block of land may be willing to pay, is the maximum of:

$$a + b800$$

subject to:

$$1.1a + 960b \le 128$$
$$1.1a + 800b \le 96$$
$$1.1a + 640b \le 80.$$

The maximum value of the objective function is now

$$\frac{1120}{11} = 101.8181..$$

attained at

$$\hat{a} = -\frac{640}{11}$$

and

$$\hat{b} = \frac{1}{5}.$$

We note from our calculations that:

$$\frac{1120}{11} < \frac{1232}{11}$$

indicating that the bid price (101.81, being what the buyer will pay) is less than the ask price (105.45, being what the seller will accept), as we would expect. (For further discussion of alternative models and scenarios, see van der Hoek and Elliott (2005), Appendix B).

A transaction will not, therefore, occur. The property (such as a block of land) will not sell until the bid price exceeds the ask price, with the benefit of the above approach being that we now know the extent of an adjustment needed in order for a transaction to be achieved, so providing a greater level of transparency.

Most papers that engage in real option valuation make assumptions that the real option is attainable (hedgeable) using tradeable underlying assets and generalize the first calculation presented here. This approach, while mathematically elegant, will not in general produce a correct value of optionality embedded in a property asset (in an economic sense), as such an asset may itself not be tradeable and its optionality not attainable by replication (cannot recreate the payoff using tradeable assets).

Relative pricing does, however, have an important role to play in developing methodologies for valuing optionality. As illustrated above, relative pricing provides a simple method of determining the bid-ask spread for a claim that is not attainable. The resulting bid and ask prices represent values of tradeable portfolios and as such provide limits, being a range within which the outcomes of alternative valuation aproaches or pricing methods can be assessed.

As any price outside the range determined by the bid-ask spread would effectively create an arbitrage opportunity, every valuation method should produce values that sit within that range. We shall illustrate that point in later sections where we introduce alternative valuation methods, indifference pricing in particular, which we argue should be used in valuing optionality embedded in property assets.

Utility-Based Pricing

As noted above, utility-based pricing uses expected utility to quantify an investor's preferences over risky alternatives. Effectively, this approach assumes that a utility value can be attached to the payoff from any investment opportunity in a way that reflects the investor's preferences and attitude toward risk. If payoffs from investment alternatives are uncertain, for example dependent on different states of the economy (such as downturn, stable and boom market), then the probabilities associated with these states are used to calculate the expected utility. The optimal investment decision is then the one that has maximum expected utility.

For our purposes, we are going to suppose that a market participant has two investment choices – being a risk free choice and a risky choice. He or she can invest all available wealth at a risk-free rate or can choose to allocate some of the funds to a risky asset (such as a vacant block of land) with the balance invested at a risk-free rate. We shall then show how the optimum bid price for that risky asset opportunity (such as a vacant block of land) may be determined using the concept of expected utility.

Let G be the non-attainable asset of the previous sub-section (the vacant block of land). We shall determine the bid price $\nu_b = \nu_b(G(1))$ of some investor with utility function U, who has wealth x at $t = 0$. The bid price solves:

$$\begin{aligned} U(xR) &= E\left[U\left((x-\nu_b)R + G(1)\right)\right] \\ &= p_1 U\left((x-\nu_b)R + G(1,1)\right) + p_2 U\left((x-\nu_b)R + G(1,2)\right) + p_3 U\left((x-\nu_b)R + G(1,3)\right). \end{aligned}$$

In other words, ν_b is chosen so that the market participant is indifferent (assigning to each the same expected utilities) between:

$$xR \quad \text{and} \quad (x-\nu_b)R + G(1)$$

at $t = 1$. Effectively, the investor is ambivalent between the two investment choices, one being a risk-free investment in cash or government bonds, the other being a bid of ν_b for a vacant block of land plus cash or government bonds.

In general, ν_b needs to be found by a numerical algorithm. If U is the exponential utility:

$$U(x) = -e^{-\lambda x}, \quad x \in \mathbb{R}$$

for $\lambda > 0$, then ν_b can be found explicitly:

$$\nu_b = -\frac{1}{\lambda R} \log\left(\sum_{j=1}^{3} p_j e^{-\lambda G(1,j)}\right) \qquad \text{Equation 6}$$

$$= -\frac{1}{1.1\lambda} \log\left(0.30e^{-\lambda 80} + 0.5e^{-\lambda 96} + 0.20e^{-\lambda 128}\right). \qquad \text{Equation 7}$$

When $\lambda = 0$,

$$\nu_b = \frac{1}{R} E[G(1)] = \frac{1}{R}\sum_{j=1}^{3} p_j \times G(1, j)$$

$$= \frac{1}{1.1}[0.30 \times 80 + 0.5 \times 96 + 0.20 \times 128]$$

$$= \frac{97.6}{1.1} = 88.7272..$$

Therefore, 88.72 represents an assesment of the bid price, being the maximum potential amount a buyer would pay for the vacant block of land, *G*, if he is *risk neutral*.

But what if the buyer is not risk neutral? We now present some numerical values for the bid prices with various choices of the risk aversion parameter λ, for the non-attainable example above and also for the attainable example with $G(1,1) = 80, G(1,2) = 96, G(1,3) = 112$.

The approximate expression used for the bid prices in Table 1 is:

$$\nu_b \cong \frac{1}{R}\left[E[G(1)] - \frac{\lambda}{2} V[G(1)] \right]$$

where $V[G(1)]$ is the variance of $G(1)$. This approximation is explained in van der Hoek and Elliott (2005), page 218. Effectively, this expression seeks to capture the extent to which, in assigning a bid price to the asset *G* (the vacant block of land), the market participant penalises the expected payoff $E[G(1)]$ for the associated risk represented by the variance. As the penalty term includes the market participant's risk aversion coefficient λ, the extent of the reduction in the bid price is expected to increase with λ. In other words, more risk market participants will bid a smaller price for G as shown in Table 1.

Risk aversion λ	0.00	0.01	0.05	0.10	0.20
Attainable ν_b	85.8181	85.2515	83.1064	80.8527	77.8964
Non-attainable ν_b	88.7272	87.5201	83.8364	80.9984	77.9009
Attainable approx	–	85.2480	82.9673	80.1164	74.4146
Non-attainable approx	–	87.4589	82.3855	76.0436	63.3600

Comparisons and Approximations of Bid Prices
Source: Authors
Table 1

A more advanced application of these ideas is used in Johnston (2004). This approach is used widely in the actuarial sciences and in general ν_b is a non-linear function of the payoff $G(1)$. For general concave utility functions $\nu_b(G(1))$ is concave in the payoff. This conforms well with the idea that there is less risk in a portfolio of payoffs and an investor is prepared to make a larger bid in this case.

The criticism of this method is that if $G(1)$ is attainable, then the bid price should be the same as the ask price. With the attainable claim of the previous section where:

$$G(1,1) = 80, \quad G(1,2) = 96, \quad G(1,3) = 112$$

the relative price is:

$$G(0) = \frac{5}{22} \times 80 + \frac{15}{22} \times 112 = \frac{1040}{11} = 94.5454\dots$$

while Equation 6 gives:

$$\nu_b = -\frac{1}{1.1\lambda} \log\left(0.3e^{-80\lambda} + 0.5e^{-96\lambda} + 0.2e^{-112\lambda}\right)$$

which produces significantly different prices for different choices of risk aversion parameter λ as shown in Table 1.

The problem that arises, therefore, is that for the attainable claim, the results given in Table 1 differ significantly from the bid price of 94.54 determined using the relative pricing approach, which should not occur. Thus, while the approach appears promising, it is not working effectively. We have thus illustrated an important shortcoming of this utility-based pricing approach – it fails to price an attainable claim correctly. For an attainable claim, the correct price is that produced by the relative pricing approach of the previous sub-section, which is the same for all market participants regardless of their risk preferences. This suggests that an improved approach should be taken.

APPLICATIONS OF CONCEPTS OF INDIFFERENCE PRICING

In this section we discuss two approaches that extend and overcome the shortcomings of the utility-based methodology described in the previous sub-section.

We begin by defining an optimal investment problem for a risk-averse market participant wishing to allocate wealth optimally between cash and a tradeable asset, such as a property index for example. We then show how to modify that investment problem to include a risky asset with built-in optionality, which could be a property asset, such as a vacant block of land.

This is done it two ways, first using stochastic discount factor pricing and then indifference pricing. We shall argue that indifference pricing is the preferred approach as it results in a methodology capable of not only pricing attainable claims correctly but also valuing non-attainable claims, producing prices with characteristics observed in practice.

An Optimal Investment Problem

Before considering a property asset, such as a vacant block of land, suppose we consider other alternatives available in the financial markets such as a property index and cash. In determining the optimum allocation of available wealth between these two alternatives, we shall again suppose that the market participant's preferences and attitude towards risk can be captured using a utility function and the optimum corresponds to maximum expected utility.

An optimal investment problem formulated in this section will be used as a basis for deriving the stochastic discounting factor approach. It will be further modified for indifference pricing.

We now find a, b with $x = a + bS(0)$, a budget constraint, so that:

$$J = E\left[U(aR + bS(1))\right]$$

$$= \sum_{j=1}^{3} p_j U(aR + bS(1, j))$$

is maximized. This is an optimal investment problem. We again present results for the exponential utility function:

$$U(x) = -e^{-\lambda x}$$

where $\lambda > 0$. In this case:

$$J = -e^{-\lambda R x} \sum_{j=1}^{3} p_j \exp\big(\lambda b(RS(0) - S(1, j))\big)$$

and so the optimal allocation $\hat{b}$ does not depend on x. The optimal allocations for various values of the risk aversion parameter are given in Table 2 along with optimal objective function value for $x = 0$.

λ	0.01	0.05	0.10	0.20
$\hat{\mathbf{b}}$	-0.8197	-0.1639	-0.0819	-0.0409
$\hat{\mathbf{J}}(0)$	0.6868	0.6868	0.6868	0.6868

Optimal Investment Problem Solutions
Source: Authors
Table 2

This shows that the risk averse market participant will invest less in the risky asset, which agrees with the usual intuition. Given non-zero values of x, we have the optimal value:

$$\hat{J}(x) = -e^{-\lambda R x} \hat{J}(0)$$

where:

$$\hat{J}(0) = \sum_{j=1}^{3} p_j \exp\big(\lambda \hat{b}(RS(0) - S(1, j))\big) = 0.6868$$

in each case, as $\lambda \hat{b} = -0.0057$ holds in each case.

We have, therefore, derived the optimal portfolios comprising only property index and cash. This may now be adjusted for the inclusion of property (such as a vacant block of land) to determine the bid price for that property included. Effectively, we seek to determine how much property index and cash an investor would be willing to give up in order to invest in property (a vacant block of land for example). As noted above, we propose to seek to determine this through two approaches, being the application of stochastic discount factor pricing and indifference pricing.

Stochastic Discount Factor Pricing

Let us start with the optimal portfolio solution derived above and consider making a small change to that portfolio to allow for an investment in asset *G* (a vacant block of land) at a bid price v_b, yet to be determined. We first give an explanation of how we could compute ν_b, and then give some example values.

From the section above, we have the following expression for the expected utility of the optimum portfolio consisting of cash and property index only:

$$\hat{J}(x) = E[U(xR + \hat{b}(S(1) - RS(0)))].$$

We now adjust this, by allocating a small fraction of available wealth away from the optimal investment in cash and tradeable asset (being property index) only, towards claim G (being a property, such as a vacant block of land). This will lead to an expression for the bid price for that claim.

Let δ be a small real number and define:

$$\hat{J}_\delta(x) = E[U((x - \delta\nu_b)R + \delta G(1) + \hat{b}(S(1) - RS(0)))].$$

If $\delta > 0$, we buy δ of the claim for price $\delta\nu_b$ and obtain the payoff $\delta G(1)$ at $t = 1$. If $\delta < 0$, this would correspond to being paid $-\delta\nu_b$ at $t = 0$ and promising to deliver $-\delta G(1)$ at $t = 1$.

If $G(1)$ is the payoff of an attainable claim, then $\hat{J}_\delta(x)$ has a maximum value when $\delta = 0$ and so:

$$\frac{\partial \hat{J}_\delta(x)}{\partial \delta}\Big|_{\delta=0} = 0.$$

This yields the formula:

$$\nu_b = \frac{1}{R}\frac{E[U'(xR + \hat{b}(S(1) - RS(0))G(1)]}{E[U'(xR + \hat{b}(S(1) - RS(0))]}. \qquad \text{Equation 8}$$

The approach of this section says that the price for a non attainable claim is also declared to be given by Equation 8. That is:

$$\nu_b = E[MG(1)]$$

where:

$$M = \frac{1}{R}\frac{\exp\left(-\lambda\hat{b}(S(1) - RS(0))\right)}{E[\exp(-\lambda\hat{b}(S(1) - RS(0)))]}$$

which does not depend on λ as $\lambda\hat{b}$ is constant (= –0.0057 in our example).

As a result this price ν_b will not depend on which exponential utility function was selected. This is known to hold also in a more general way (Davis 1997). In our example:

$$M(1) = 0.185067$$

$M(2) = 0.687010$
$M(3) = 2.550329$

(where the three possible values correspond to three possible states of the world at time t = 1 – for example, downturn, stable and boom market) and:

$$v_b = \sum_{j=0}^{3} p_j M(j) G(1, j).$$

For the attainable claim:

$$G(1,1) = 80, \quad G(1,2) = 96, \quad G(1,3) = 112$$

we have $v_b = 94.5454$ which accords with the assessment derived previously by relative pricing, and for a non-attainable claim:

$$G(1,1) = 80, \quad G(1,2) = 96, \quad G(1,3) = 128$$

$v_b = 104.7468$ which is within the range of 101.81 bid price and 105.45 ask price derived previously by relative pricing.

This pricing methodology has been proposed for real options and could be applied in the context of property investments. For examples of the valuation of non-market securities, see Williams (1993), Flores (2003), Johnston (2004), Henderson (2005), Miao and Wang (2007) or Edge (2011).

However, while a stochastic discount pricing method prices non-attainable claims and produces correct prices for attainable claims, it leads to linear pricing. While desirable in other contexts, in property valuation linear pricing may not adequately reflect risk characteristics of assets. While linear pricing may correctly value two assets in isolation, it may not provide an assessment of the synergistic benefits of having both in combination, such as in a diversified portfolio.

Consider a portfolio of property investments. With linear pricing, the value of the portfolio will simply be the sum of the values of assets comprising that portfolio. However, there is, in principle, a risk diversification benefit from holding a portfolio of property assets and so a risk-averse market participant may assign a bid to the portfolio that is higher than the sum of its parts. In other words, the pricing function may be concave.

Indifference Pricing

An alternative approach, that addresses issues associated with linear pricing of non-attainable claims, is *indifference pricing* which may be adopted, for example, to explain why a portfolio of properties may transact for more than the sum of the individual values of the separate properties. As we shall argue, *indifference pricing* is a method which bridges the gap between pricing in financial markets and pricing non-market securities in a consistent way.

For an indifference approach, we continue to work with expected utilities and compare a portfolio of property index and cash with a portfolio of property index, cash and property (such as a vacant block of land). The bid price v_b attributable to the property (being the vacant block of land) is then the amount that makes the expected utility the same for both investment choices. In other words, the bid price will now be the amount that makes the market participant ambivalent between investing only in tradeable financial assets (cash and property index) and liquidating some of the financial asset holdings in order to invest in property (the vacant block of land).

Therefore, in order to determine the indifference bid price ν_b of the claim $G(1)$ at $t=1$, we first proceed as above and obtain:

$$\hat{J}(x) = E[U(xR + \hat{b}(S(1) - RS(0))].$$

representing the portfolio comprising property index and cash. We also maximize:

$$E[U((x-\nu_b)R + G(1) + b(S(1) - RS(0)))] \qquad \text{Equation 9}$$

(representing the portfolio comprising property index, cash and property) over b. We use the budget constraint:

$$a + bS(0) = x - \nu_b$$

and instead of obtaining $aR + bS(1)$ we obtain:

$$aR + bS(1) + G(1).$$

Then the optimum in Equation 9 is:

$$\hat{J}_G(x-\nu_b) = E[U((x-\nu_b)R + G(1) + \tilde{b}(S(1) - RS(0)))]$$

where $\tilde{b}$ is the new optimizer. We determine the bid price ν_b so that we are indifferent between:

$$(x-\nu_b)R + G(1) + \tilde{b}(S(1) - RS(0))$$

and:

$$xR + \hat{b}(S(1) - RS(0)).$$

The equation:

$$\hat{J}_G(x-\nu_b) = \hat{J}(x) \qquad \text{Equation 10}$$

then determines the indifference bid price which we write:

$$\nu_b = \nu_b(G(1)).$$

In a similar way, we can define an indifference ask price $\nu_a(G(1))$ and it satisfies:

$$\nu_a(G(1)) = -\nu_b(-G(1)).$$

We now show how this works with exponential utilities and obtain some answers for the previous examples. For exponential utilities, Equation 9 becomes:

$$-exp\left(-\lambda(x-v_b)R\right)\sum_{j=1}^{3} p_j\ exp\left(-\lambda\left[G(1,j)+b(S(1,j)-RS(0))\right]\right)$$

which we maximize for b for various choices of λ. In fact, we minimize:

$$\sum_{j=1}^{3} p_j\ exp\left(-\lambda\left[G(1,j)+b(S(1,j)-RS(0))\right]\right)$$

with respect to b, to give $\tilde{b}$ which we tabulate for various choices of λ in Tables 3 and 4.

We shall do this in two cases, for the non-attainable claim $(G(1,1),\ G(1,2),\ G(1,3)) = (80,\ 96,\ 128)$ and for the attainable claim $(80,\ 96,\ 112)$.

Non-Attainable Claim:
We have:

$$\hat{J}_G(x-v_b) = -exp\left(-\lambda R(x-v_b)\right)\tilde{J}_G(0)$$

where:

$$\tilde{J}_G(0) = \sum_{j=1}^{3} p_j\ exp\left(-\lambda\left(G(1,j)+\tilde{b}(S(1,j)-RS(0))\right)\right).$$

Table 3a provides optimal values for various choices of risk aversion parameters λ.

λ	0.01	0.05	0.10	0.20
$\tilde{\mathbf{b}}$	-1.0151	-0.3641	-0.2876	-0.2876
$\tilde{\mathbf{J}}_{\mathbf{G}}\mathbf{(0)}$	0.2171	2.195×10^{-3}	7.188×10^{-6}	9.131×10^{-11}

New Optimizers by Risk Aversion Level for a Non-Attainable Claim
Source: Authors
Table 3a

We now compute v_b so that Equation 10 holds. This means that:

$$exp\left(-\lambda Rx\right)\hat{J}(0) = exp\left(-\lambda R(x-v_b)\right)\tilde{J}_G(0).$$

The optimizers for the left hand side were obtained in Table 2.

We then have:

$$v_b = \frac{1}{\lambda R}\log\left[\frac{\hat{J}(0)}{\tilde{J}_G(0)}\right].$$

Prices corresponding to different levels of risk aversion are given in Table 3b. For comparison, the Table also includes prices from the previous section calculated using the pricing Equation 8 derived previously.

λ	0.01	0.05	0.10	0.20
$\tilde{J}(0)$	0.6868	0.6868	0.6868	0.6868
$\tilde{J}_G(0)$	0.2171	2.195×10^{-3}	7.188×10^{-6}	9.131×10^{-11}
v_b	104.6846	104.4620	104.2485	103.3679
Eq. 8	104.7468	104.7468	104.7468	104.7468

Comparison of Bid Prices by Risk Aversion Level for a Non-Attainable Claim
Source: Authors
Table 3b

We note that all of the values of ν_b (being 103.37 to 104.68) in Table 3b lie within the range from 101.81 (bid) to 105.45 (ask) derived previously by relative pricing. The results in Table 3b also indicate that increasing the risk aversion coefficient λ decreases ν_b away from the ask (105.45) towards the bid (101.81).

Attainable Claim

However, we now repeat similar calculations using the same formulas for an attainable claim with the results shown in Table 4.

λ	0.01	0.05	0.10	0.20
$\tilde{b}$	-0.9197	-0.2639	-0.1819	-0.1409
$\tilde{J}_G(0)$	0.2427	3.788×10^{-3}	2.090×10^{-5}	6.360×10^{-10}
v_b	94.5454	94.5454	94.5454	94.5454

Comparison of Bid Prices and Optimizers by Risk Aversion Level for an Attainable Claim
Source: Authors
Table 4

It should be noted that the value of ν_b in Table 4 is the same for all risk aversion levels (94.54), which is the value that was previously determined using relative pricing. The attainable claim has thus been priced correctly.

With indifference pricing, it can be shown that the indifference price of an attainable claim is always the same as the financial (relative) price for any utility function U, not just for the exponential utilities.

These methods have been presented in more advanced form in Davis (1997), Zariphopoulou (2001), Carmona (2009), Elliott and van der Hoek (2009), Henderson and Hobson (2009), Alexander and Chen (2012) and in references cited therein.

In general, the indifference bid price $\nu_b(G)$ has the following four characteristics (see van der Hoek and Elliott (2005), Chapter 14 for a simple presentation):

(1) If $G = G_1 + G_2$ where G_1 is attainable (such as, for example, a vacant block of land comprised of a hedgeable and non-hedgeable element), then:

$$\nu_b(G) = \pi(G_1) + \nu_b(G_2)$$

where $\pi(G_1)$ is the usual relative price as explained in the previous section.

(2) If G_1, G_2 are two claims and $0 \le \alpha \le 1$ then:

$$\nu_b(\alpha G_1 + (1-\alpha)G_2) \ge \alpha \nu_b(G_1) + (1-\alpha)\nu_b(G_2)$$

(3) If $H^- \le G \le H^+$ where $H^\pm$ are attainable claims, then:

$$\pi(H^-) \le \nu_b(G) \le \pi(H^+).$$

(4) If $G_1 \le G_2$ then:

$$\nu_b(G_1) \le \nu_b(G_2).$$

These four characteristics of the indifference bid price can be explained intuitively as follows:

Characteristic 1 says that if a claim (an investment opportunity) can be decomposed into two components, one that is attainable (can be replicated using cash and other tradeable securities) and another that is not attainable (reflecting for example some risks unique to that investment opportunity), then the indifference pricing approach will produce the correct relative price for the attainable part of the claim and the optimum bid price for the non-attainable part.

Characteristic 2 indicates that the indifference price is concave, which means in particular that the indifference pricing methodology will, if appropriate, produce the bid price for a portfolio of property assets that is higher than the sum of bid prices of the individual assets in that portfolio. It is thus capable of assigning value to the added benefit of diversification when assets are held in a portfolio.

Characteristic 3 is equivalent to saying that the indifference price of a non-attainable claim will always lie within the bid-ask spread for that claim, determined using relative pricing as shown in the previous section.

Finally, characteristic 4 indicates that if a risky investment promises an equal or better return compared to some other risky opportunity, then its indifference bid price will be at least as high as the indifference bid price of that other opportunity, as would normally be expected.

We may summarise, therefore, that in this way, indifference pricing provides a bridge between financial pricing and actuarial pricing where claims cannot be hedged in financial markets.

It would, however, be interesting, in real property options, to explore the indifference price concept using a property index as the asset S.

Areas for Further Research

While this paper has used the expected utility paradigm, there are other methodologies for ranking investment opportunities. Another approach would be to study valuation in terms of transfer of risk between market participants, with recent studies with risk measures leading to significant advances in this context (see, for example, Föllmler and Schied (2004) and Barrieu and El Karoui (2009) who provide insights into the use of risk measures in real option valuation which may present a more tractable approach to indifference pricing without the use of utility functions).

Regardless of the valuation method used, consistency across multiple periods should hold. For example, if we have times $t = 0, 1, 2$ and a claim is paid at $t = 2$, its value at $t = 0$ can be found in two ways. We could view the valuation as a two time problem using $t = 0, 2$ or we could find the value of the claim at $t = 1$ and then find the value of this result at $t = 0$. One would require that the two answers should agree. This is usually called *time consistency in valuation*. Dynamic risk measures have been developed to address this issue. Connection with valuation is explained in Barrieu and El Karoui (2004), which also provides an extension of indifference pricing outside of the expected utility theory setting.

At present, however, the expected utility framework or its approximations provide the most tractable approaches.

CONCLUSIONS

This paper sought to apply concepts of optionality and indifference pricing to explore how a property asset, such as the vacant block of land considered by Titman (1985), might be valued in an inactive property market devoid of property transactions by considering it as an alternative to investing in other, non-property asset classes, with market participants' preferences and attitudes towards risk being modelled using the concept of utility.

We have argued that property assets, such as the vacant block of land considered in Titman (1985), should be viewed as non-attainable claims, namely claims whose payoffs cannot be recreated using other traded assets (cash and a property index in our examples). This makes the application of methods traditionally applied to value options on financial assets problematic. As shown above, the relative pricing approach of Black and Scholes can only be used to obtain a range of prices within which the value of the vacant block should lie, to prevent arbitrage oppotunities. This range is determined by the bid price (what the buyer will pay) and the ask price (what the seller will accept), with the bid price being lower than the ask price. While this type of range would indicate that a transaction may not actually occur, it is nonetheless useful in that it provides a benchmark against which alternative valuation approaches or pricing methods can be assessed. For the numerical example used in the paper, this price range for a non-attainable claim was from 101.81 to 105.45. We also considered an attainable claim for which the relative pricing approach produced a single (correct) value of 94.54.

Generally, we would expect a good valuation approach to produce a value for a vacant block of land that is within the range determined by relative pricing. Moreover, we would expect such an approach to be capable of pricing purely financial claims correctly. We have shown above that a utility-based approach, while attractive in that it takes into account individual investor risk preferences through the use of a utility function, can produce values for a non-attainable claim outside of the required range (values from 63.36 to 87.45 for our numerical example) and also fail to price an attainable claim correctly (a range of different values depending on the risk aversion parameter instead of 94.45 in all cases).

The stochastic discount factor pricing approach then discussed proved to be an improvement in that it produced the correct value of 94.45 for the attainable claim, and a bid price for a non-attainable claim (104.75) within the required range. However, the stochastic discount factor approach leads to linear pricing, and as such it is not able to provide an assessment of benefits from synergies that may exist within portfolios of property assets.

As illustrated thereafter, the indifference pricing approach overcomes the shortcomings of the other approaches considered in this paper. Through an application of indifference pricing with differing levels of risk aversion we determined a bid price range of 103.36-104.68 for a non-attainable claim (which is within the range of 101.81-105.45 derived previously by relative pricing) and 94.54 for an attainable claim (which accords with that determined by relative pricing).

Overall, indifference pricing provides a correct analytical framework for valuation by considering how a market participant might choose between equivalent investment alternatives. Therefore, in the absence of property transactions from which to assess the value of a property, such as a vacant block of land which is a non-attainable claim, indifference pricing emerges as an approach that may be adopted in such circumstances.

This paper did not seek to prescribe a series of formulae for immediate application by the property valuation profession, but rather sought to raise issues for consideration and discussion by property valuation academics and practitioners as a contribution to the evolution of the property valuation discipline.

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